ILLC Project Course in Statistical Learning Theory

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Random Products

Problem

You enter the casino with a capital of $C_0 = 1$. In each round,

1. your capital doubles with 75% probability, and
2. decreases by a factor of 10 with 25% probability.

Is this good news? What will happen after $t = 1000$ games?

$$C_{1000} \ = \ F_1 \times F_2 \times F_3 \times \cdots \times F_{1000}.$$

e.g.,

$$C_{1000} \ = \ 2 \times 2 \times \frac{1}{10} \times 2 \times \cdots \times \frac{1}{10} \times 2.$$

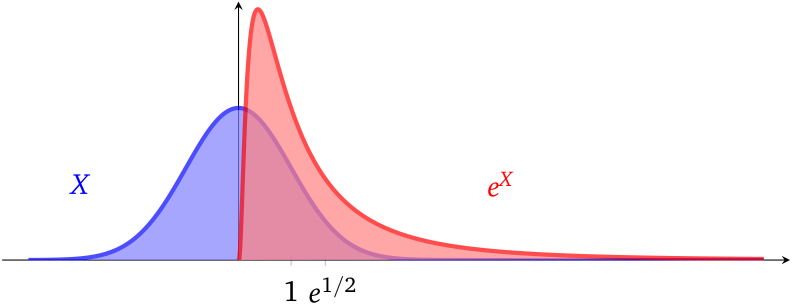
Moment-Generating Functions

Definition

The **moment-generating function** of a random variable X is

$$G(r) = E\left[e^{rX}\right]$$

whenever these expected values are well-defined.



Moment-Generating Functions

Definition

The **moment-generating function** of X is

$$G(r) = E\left[e^{rX}\right]$$

whenever this is well-defined.

Example

Suppose $X \sim \text{Bernoulli}(p)$, i.e.,

x	1	0
$\Pr(X = x)$	p	$1 - p$

What is $G(r)$?

Moment-Generating Functions

Example

Suppose $X \sim \text{Geometric}(1/2)$, i.e.,

x	1	2	3	4	$\cdots$
$\Pr(X = x)$	$1/2$	$1/4$	$1/8$	$1/16$	$\cdots$

What is $G(r)$?

(cf. `en.wikipedia.org/wiki/St._Petersburg_paradox`)

Exponential Bounds

Problem

Let

$$S_t = X_1 + X_2 + \cdots + X_t$$

be a sum of i.i.d. variables.

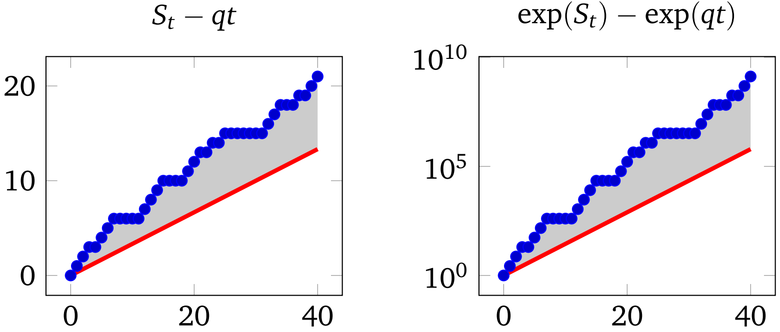
Which of the following two series diverge the fastest?

$$S_1, S_2, S_3, S_4, \ldots \qquad \text{from} \qquad q, 2q, 3q, 4q, \ldots$$

or

$$e^{S_1}, e^{S_2}, e^{S_3}, e^{S_4}, \ldots \qquad \text{from} \qquad e^{q}, e^{2q}, e^{3q}, e^{4q}, \ldots$$

Exponential Bounds



Exponential Bounds

Theorem (the exponential Markov bound)

$$\Pr\{S_t > qt\} \leq \left(\frac{G(r)}{e^{rq}}\right)^t.$$

Proof.

The Markov bound plus the fact that

$$G_{S_t}(r) = G_X(r)^t.$$

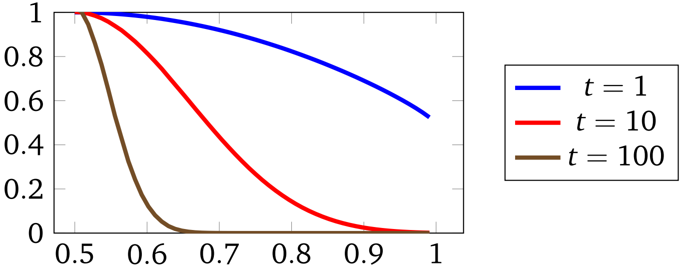
□

Exponential Bounds

Theorem (The Chernoff Bound)

For a sum of t Bernoulli variables with parameter p,

$$\Pr\left\{\frac{S_t}{t} > q\right\} \;\leq\; \left(\left(\frac{p}{q}\right)^q \left(\frac{1-p}{1-q}\right)^{1-q}\right)^t .$$

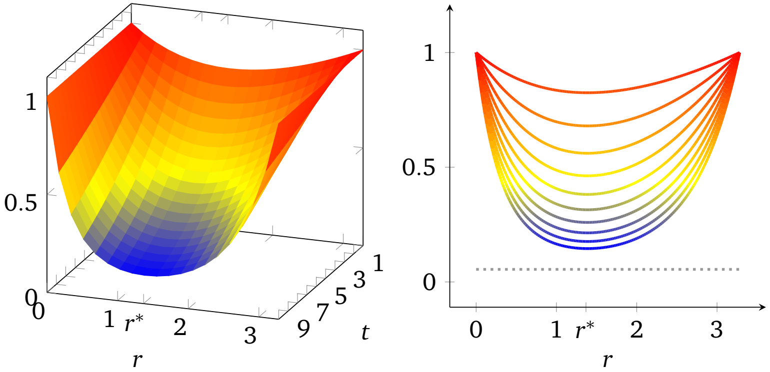


Exponential Bounds

Proof.

$$\Pr\left\{\frac{S_t}{t} > q\right\} \leq \left(\frac{pe^r + 1 - p}{e^{rq}}\right)^t.$$

□

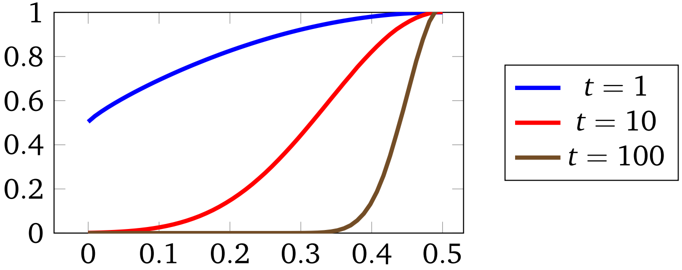


Exponential Bounds

Theorem (The reverse Chernoff Bound)

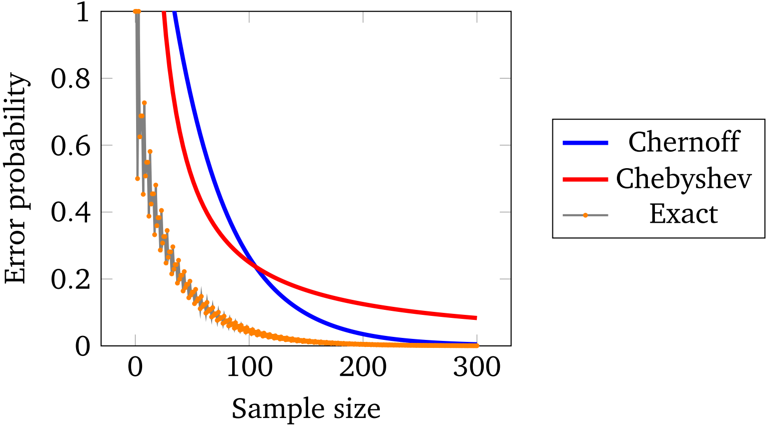
For a sum of t Bernoulli variables with parameter p,

$$\Pr\left\{\frac{S_t}{t} < q\right\} \le \left(\left(\frac{p}{q}\right)^q \left(\frac{1-p}{1-q}\right)^{1-q}\right)^t .$$



Exponential Bounds

$$\Pr\left\{\left|\frac{S_t}{t} - 0.5\right| > 0.1\right\}$$



Exponential Bounds

Theorem (the unit interval Hoeffding bound)

All of the previous theorems are also true for arbitrary bounded variables with $0 \leq X \leq 1$ and $E[X] = p$.

Theorem (the general Hoeffding bound)

For a sum of bounded variables with $a \leq X \leq b$ and $E[X] = p$,

$$\Pr\{S_t > qt\} \leq \left(\left(\frac{p-a}{q-a}\right)^{q-a} \left(\frac{b-p}{b-q}\right)^{b-q} \right)^{t/(b-a)} .$$