ILLC Project Course in Statistical Learning Theory

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The VC Bound

Theorem

Let

$$t^* = \sup t \qquad \textit{subject to} \qquad N(t) = 2^t.$$

If $t^* < \infty$, then $N(t) \le \Phi(t^*, t)$ for all t.

Proof.

Define $S_t := (S \downarrow x)$, $S_{t+1} := (S \downarrow x, x_{t+1})$, and

$$R := \{ A \in S_t \mid A \in S_{t+1} \wedge A \cup \{x_{t+1}\} \in S_{t+1} \}.$$

Then

$$\begin{aligned} N_S(t+1) &= N_S(t) + N_R(t) \\ &\le \Phi(t^*, t) + \Phi(t^* - 1, t) \\ &= \Phi(t^*, t+1). \end{aligned}$$

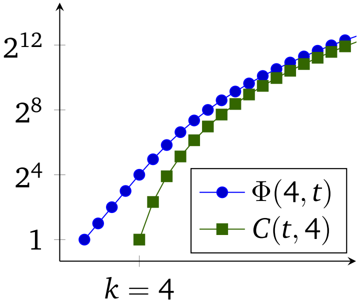
□

The VC Bound

$$N(t) \leq \binom{t}{0} + \binom{t}{1} + \binom{t}{2} + \cdots + \binom{t}{t^*}$$

The VC Bound

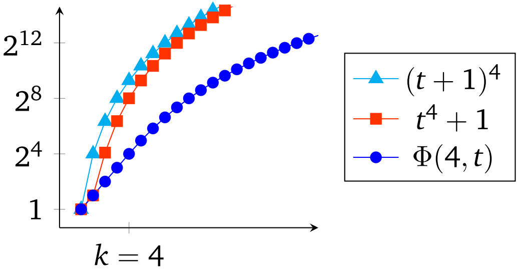
$$\Phi(k,t) = \binom{t}{k} + \binom{t}{k-1} + \binom{t}{k-2} + \cdots + \binom{t}{0}$$



The Combinatorical Bound

Theorem

$$\Phi(k,t) \;\le\; t^k + 1 \;\le\; (t+1)^k$$

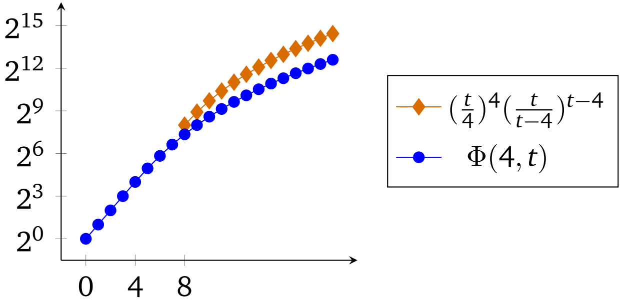


The Entropy Bound

Theorem

For $t \geq 2k$,

$$\Phi(k,t) \;\leq\; \left(\frac{k}{t}\right)^k \left(\frac{t-k}{t}\right)^{t-k} \;=\; \exp\left(-t H_2\left(\frac{k}{t}\right)\right)$$



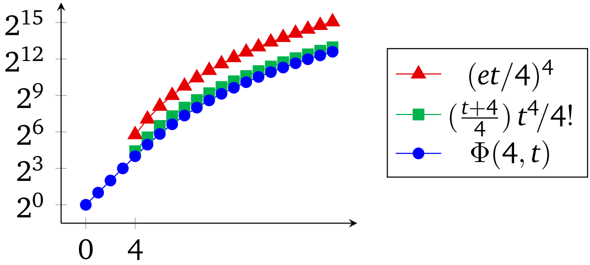
The Recursive Bound

Theorem

For $t \geq k$,

$$\Phi(k,t) \;\leq\; \frac{(t+k)t^{k-1}}{k!} \;\leq\; 2\frac{t^k}{k!} \;\leq\; \left(\frac{et}{k}\right)^k$$

where, by convention, we evaluate the bound as 1 *for* $k = 0$.



The Recursive Bound

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$t = 0$	2	–	–	–	–	–
$t = 1$	2	2	–	–	–	–
$t = 2$	2	4	4	–	–	–
$t = 3$	2	6	9	9	–	–
$t = 4$	2	8	16	21.3	21.3	–
$t = 5$	2	10	25	41.7	52.1	52.1

The Recursive Bound

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$t=0$	1	1	1	1	1	1
$t=1$	1	2	2	2	2	2
$t=2$	1	3	4	4	4	4
$t=3$	1	4	7	8	8	8
$t=4$	1	5	11	15	16	16
$t=5$	1	6	16	26	31	32

The Recursive Bound

Proof.

Recursion step:

$$\Phi(k+1, t+1) \;\leq\; 2\frac{t^{k+1}}{(k+1)!} + 2\frac{t^k}{k!} \;=\; 2\frac{t^{k+1}}{(k+1)!}\left(1 + \frac{k+1}{t}\right).$$

An estimation:

$$\left(\frac{t+1}{t}\right)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} \left(\frac{1}{t}\right)^i \;\geq\; 1 + \frac{k+1}{t}.$$

□